

Approximating the Euclidean Circle in the Square Grid using Neighbourhood Sequences

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Abstract

Distance measuring is a very important task in digital geometry and digital image processing. Due to our natural approach to geometry we think of the set of points that are equally far from a given point as a Euclidean circle. Using the classical neighbourhood relations on digital grids, we get circles that greatly differ from the Euclidean circle.

In this paper we examine different methods of approximating the Euclidean circle in the square grid, considering the possible motivations as well. We compare the perimeter-, area-, curve- and noncompactness-based approximations and examine their realization using neighbourhood sequences.

We also provide a table which summarizes our results, and can be used when developing applications that support neighbourhood sequences.

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1 Introduction

The classical digital geometry started with [1], where the authors defined the two basic neighbourhood relations on the square grid. The topic is well developed due to people of image processing and computer graphics communities. In [2] the authors used the so-called neighbourhood sequences to vary the neighbourhood criterion in a path. They used only periodic neighbourhood sequences in their analysis. Some properties of distances based on neighbourhood sequences are detailed in [3]. The concept of neighbourhood sequences was extended to not necessarily periodic sequences in [4]. These general neighbourhood sequences were analysed in [5, 6], formulae to compute

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distances are presented in [7]. One of the main problems of digital geometry is the approximation of the Euclidean circle (for instance in [8]). It is the topic of the present paper as well.

After the formal definitions (Section 2) some previous results on the topic are recalled (Section 3). The approximations of the circle by neighbourhood sequences (i.e. the digital discs) are octagons. The descriptors of these octagons are presented in Section 4. In Section 5 several approaches of approximation are detailed, in Section 6 a detailed example is shown, while in Section 7 we summarize the results. Finally a conclusion closes the paper.

2 Definitions

Our aim is to provide the best approximation of a circle with a given radius using neighbourhood sequences. In order to formulate our results, we have to recall the following definitions [2, 4, 8]. Because a circle is a planar shape, we restrict our considerations to two dimensions.

Definition 1. *Let p and q be two points in \mathbb{Z}^2 and $j \in \{1, 2\}$. The i th coordinate of the point p is indicated by $Pr_i(p)$. The points p and q are j -neighbours in two dimensions if the following two conditions hold:*

- $|Pr_i(p) - Pr_i(q)| \leq 1 \quad (\forall i \in \{1, 2\}),$
- $\sum_{i=1}^2 |Pr_i(p) - Pr_i(q)| \leq j.$

1-neighbourhood corresponds to the classical 4-neighbourhood and 2-neighbourhood to 8-neighbourhood, as it can be seen on Fig. 1 left and right, respectively. In two dimensions, j -neighbourhood means that we can step along at most j Cartesian axes to reach a neighbour.

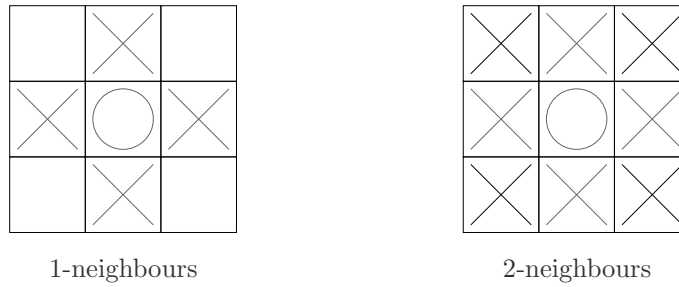


Figure 1: The X-marked points are j -neighbours of the O-marked one

In a neighbourhood sequence one can give a sequence of neighbourhoods, in which a j -neighbourhood is represented by the number j . The i th element of such a sequence defines the neighbourhood we can use as we take the i th step going further and further

from the starting point. Formally in the n dimensional digital space the neighbourhood sequences are defined in the following way:

Definition 2. The infinite sequence $B = (b_i)_{i=1}^{\infty}$, where $b_i \in \{1, \dots, n\}$ for all $i \in \mathbb{N}$, is called a generalized nD -neighbourhood sequence.

In \mathbb{Z}^2 , neighbourhood sequences are infinite sequences of values 1 and 2. Now we can define the path leading from one point to another, its length, and the distance between two points.

Definition 3. Let p and q be two points in \mathbb{Z}^2 and B a generalized $2D$ -neighbourhood sequence. The point sequence $\Pi(p, q; B)$ - which has the form $p = p_0, p_1, \dots, p_m = q$, where p_{i-1} and p_i are b_i -neighbours for $i \in \{1, 2, 3, \dots\}$ - is called a path from p to q determined by B . The length $|\Pi(p, q; B)|$ of the path $\Pi(p, q; B)$ is m .

Definition 4. Let p and q be two points in \mathbb{Z}^2 and B a generalized $2D$ -neighbourhood sequence. The/a shortest path from p to q is denoted by $\Pi^*(p, q; B)$. The distance between p and q is defined as the length of the minimal path, and is written as

$$d(p, q; B) = |\Pi^*(p, q; B)|.$$

As one can see, neighbourhood sequences occupy digital octagons after every step (see Fig. 2). This means, that if we use neighbourhood sequences on the square grid, the only shape we can use to approximate circles are special octagons, having 135° inner angles [8], or squares with right angles in degenerated cases. The octagon is degenerated if only 1 type of neighbourhood is used to generate it. On Fig. 2, we can see two discs with a radius of 4, generated by the neighbourhood sequences $B = (1, 1, 2, 1, \dots)$ and $B = (2, 2, 2, 1)$, respectively. On the figures, each point of the digital discs is indicated by a number, which is the distance of the point from the center point with the given sequence.

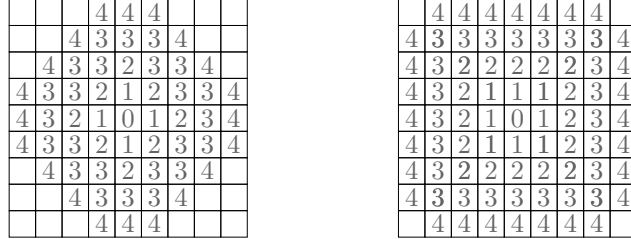


Figure 2: Examples of digital discs with a radius of 4

3 Previous results

In [8] Hajdu and Nagy used the isoperimetric ratio ([2]) to approximate the Euclidean circle.

Definition 5. We call

$$\kappa := \frac{P^2}{A},$$

the isoperimetric ratio of the shape, where P is the perimeter, and A is the area of the shape.

Remark. In [2] and [8] the isoperimetric ratio is referred as *noncompactness ratio*.

In Euclidean geometry, according to the isoperimetric inequality, the isoperimetric ratio is never less than 4π , and is minimal for the circle. Let k_1 denote the number of value 1 in the neighbourhood sequence B , and k_2 the number of value 2. The number of steps we take in the neighbourhood sequence B will be denoted by k , and $k = k_1 + k_2$. Let c denote the proportional frequency of value 2 in the sequence, thus $c := \frac{k_2}{k} = \frac{k_2}{k_1 + k_2}$. The isoperimetric ratio for the convex hull of the possible octagons can be written in the form

$$\kappa(c) = 16 \left(1 - 2 \left(2 - \sqrt{2} \right) \frac{c(1-c)}{c(2-c) + 1} \right).$$

It turns out that $\kappa(c)$ is minimal at $c = \sqrt{2} - 1$. If c satisfies this equality, the convex hull of the pixel centers becomes a regular octagon, thus the problem of approximating the circle turns out to be the problem of approximating the regular octagon. Note that because of the irrationality of $\sqrt{2} - 1$, c can only be equal to this in the ideal case of $k \rightarrow \infty$. For any $\varepsilon > 0$ there is a value of k_0 such that for any $k > k_0$ $\exists k_2 : \frac{k_2}{k} - (\sqrt{2} - 1) < \varepsilon$.

Let us remark that the use of the isoperimetric ratio can be avoided. Taking into consideration the connection between the isoperimetric ratio and the isoperimetric problem, we can say that Hajdu and Nagy solved the isoperimetric problem for the corresponding octagons in their approximations on the square grid. The proof is much easier if we solve the isoperimetric problem directly for these octagons. Further in this article we take into account the circle's isoperimetric property by restricting our examinations only to regular octagons. This can only be achieved if we assume that the approximating octagon is large enough to have more than one value 1 and 2 in its sequence ($k_1, k_2 \geq 1$).

Hajdu and Nagy also gave the sequence which in every step generates the octagon closest to the regular octagon. In this paper we use a different approach. We assume that we have a circle with a particular radius r . We want to determine the sequence that leads to the best approximating regular octagon. We can also approximate circles with positive real radius, which could not be done by using the old approach. We use different measures to describe the generated octagons, which we will call '*descriptors*'. We also investigate several possible definitions for 'approximation'.

4 Descriptors of the octagons

The sequence-generated octagons can be easily described by their sidelengths a and b (see Fig. 3). We use three types of measure to give the length of the sides: the pixel,

the inner convex hull (or inner octagon) and the outer convex hull (or outer octagon) based descriptors.

Pixel based descriptors: a and b are given in pixels. As it can be seen on Fig. 3, we assume that corner pixels belong to the horizontal sides (a).

Inner octagon: in this case, a and b are equal to the corresponding sides of the convex hull of the centers of the pixels.

Outer octagon: the same as the inner octagon, except that we use the convex hull of the pixels as squares.

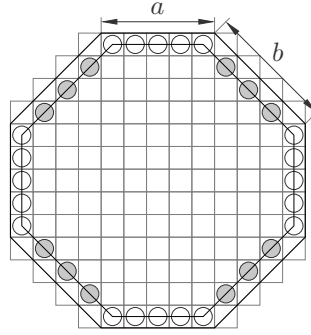


Figure 3: Descriptors

Table 1 shows the values for the sides, the perimeter and the area of the octagons.

	pixel ($k_1 \geq 1$)	inner hull	outer hull
a	$2k_2 + 1$	$2k_2$	$2k_2 + 1$
b	$k_1 - 1$	$\sqrt{2}k_1$	$\sqrt{2}k_1$
P	$4(a + b)$	$4(a + b)$	$4(a + b)$
A	$(2k_2 + 1)^2 + 2k_1(4k_2 + k_1 + 1)$	$a^2 + 2\sqrt{2}ab + b^2$	$a^2 + 2\sqrt{2}ab + b^2$

Table 1. Attributes of the octagons

In the following section the inner and outer octagons give almost the same results (and they are equal to each other in the ideal case $k \rightarrow \infty$), so we omit the consideration of the outer octagon.

If our aim is only to make the octagon of the sequence the best approximating one in the sense that it is isoperimetric (it is as regular as it can be after a large fixed k number of steps), we only need to solve the equation $a = b$ for each descriptors in the case of $k \rightarrow \infty$. Let p denote the probability of the occurrence of value 2 in the neighbourhood sequence, i.e.,

$$p := \lim_{k \rightarrow \infty} \frac{k_2}{k}.$$

Using this notation, we get that the regular octagons for large distances ($k \rightarrow \infty$) occur at $p = \frac{1}{3}$ when using the pixel-based descriptors and at $p = \sqrt{2} - 1$ for the inner and outer convex hulls, just as in [8].

In [8], the concept of ‘convex hull’ is equivalent to the descriptor we call inner octagon.

5 Possible ways of approximating the circle

In this section we split our ideas by answering the question: in what sense would we like to approximate the circle? We can use different measures to order the approximating octagons from ‘badly’ approximating octagons (far from the result) to ‘well’ approximating ones (close to the result). In this terminology we would like to minimize a distance function between the circle and the approximating shapes.

By inverting the results in Table 1 and using $a = b$, we can give the number of values 1 and 2 as a function of the sidelength a (see Table 2). This way we also get the minimal length $k = k_1 + k_2$ of the sequence belonging to the best approximating regular octagon. In Table 2 $[.]$ denotes the rounding function.

	pixel ($k_1 \geq 1$)	inner octagon	outer octagon
k_1	$[a + 1]$	$\left[\frac{a}{\sqrt{2}}\right]$	$\left[\frac{a}{\sqrt{2}}\right]$
k_2	$\left[\frac{a-1}{2}\right]$	$\left[\frac{a}{2}\right]$	$\left[\frac{a-1}{2}\right]$
p	$\frac{1}{3}$	$\sqrt{2} - 1$	$\sqrt{2} - 1$

Table 2. Number of values 1 and 2 as a function of sidelength a

This means that we only need to compute a in order to gain the appropriate sequence. In the following subsections we only provide the formulas from which we formulated our results.

Perimeter based approximation

We would like to get the octagon having the same perimeter as the given circle. Formally we have to solve the following equation for a_{perim} :

$$P_{regoct} = 8a = 2\pi r$$

$$a = \frac{\pi}{4} r$$

We get different results using different descriptors. For each approach we need to get the ideal (continuous) length of a_{ideal} (in this case a_{perim}), then solve the equations $a_{ideal} = a$ and $a_{ideal} = b$ by substituting the formulas of a and b from Table 1. This way we get the sequence of the best approximating regular octagon by having the values of k_1 and k_2 . We follow the same method in all cases.

Area based approximation

Let us determine the regular octagon with the same area as the given circle by solving the following equation for a :

$$A_{regoct} = 2 \left(1 + \sqrt{2} \right) a^2 = \pi r^2$$
$$a = \sqrt{\frac{\pi}{2 \left(1 + \sqrt{2} \right)}} r$$

Inscribed circle based approximation

The radius based approximations use the concept of the radius of the regular octagon, which is the distance between the center of the octagon and the sides, which is the same as the radius of the inscribed circle of the octagon. There are two types of radius based approximation: the inscribed circle based and the covering circle based method.

In the inscribed circle based approximation we would like to get the sequence that generates the regular octagon which is closest to the octagon having the given circle as its inscribed circle.

Since r denotes the given radius, in this case it is the same as the radius of the generated octagon. We have to solve the following equation (for notions see Fig. 4):

$$\tan \frac{\pi}{8} = \sqrt{2} - 1 = \frac{a}{2r}$$
$$a = 2 \left(\sqrt{2} - 1 \right) r$$

Covering circle based approximation

We need the sequence generating the octagon having the given circle as its covering circle. This means that the radius of the given circle on Fig. 4 is denoted by R .

$$R^2 = \left(\frac{a}{2} \right)^2 + r^2$$
$$a = \sqrt{2 - \sqrt{2}} R$$

Least squares difference

Finally, we discuss curve based approximations. In these methods we use curve fitting, i.e., we search the regular octagon having its curve the closest to the curve of the given circle. By defining the distance between the two curves differently, we get two methods: the least square and the least sum of distances approximations. Due to symmetric reasons we only need to examine a fraction (namely one sixteenth) of the curves. To

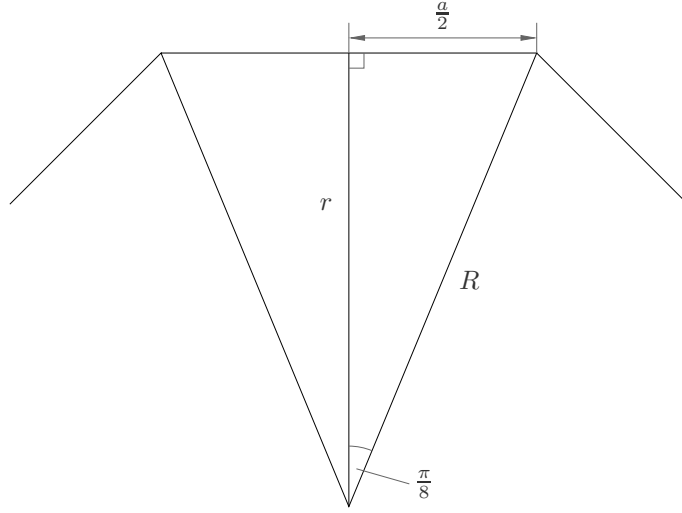


Figure 4: Radius based approximation

make the integration easier we convert the problem into a planar polar coordinate system (see Fig. 5), in which we denote the curve of the circle by $z_0(\alpha)$ and the curve of the octagon by $z(\alpha)$:

$$z_0(\alpha) = r \quad z(\alpha) = \frac{m}{\cos \alpha}.$$

Using the least square method the distance between the two curves at α becomes

$$f(\alpha, r) := (z(\alpha) - z_0(\alpha))^2.$$

The distance of the two curves can be determined by integrating these elementary distances:

$$F(r) := \int_0^{\frac{\pi}{8}} f(\alpha, r) d\alpha.$$

This way the search for the closest octagon becomes an extremum problem for $F(r)$. By solving $F'(r) = 0$ we get the optimal sidelength a , which is

$$a = \frac{\pi}{4(\sqrt{2} + 1) \ln \tan\left(\frac{5\pi}{16}\right)} r.$$

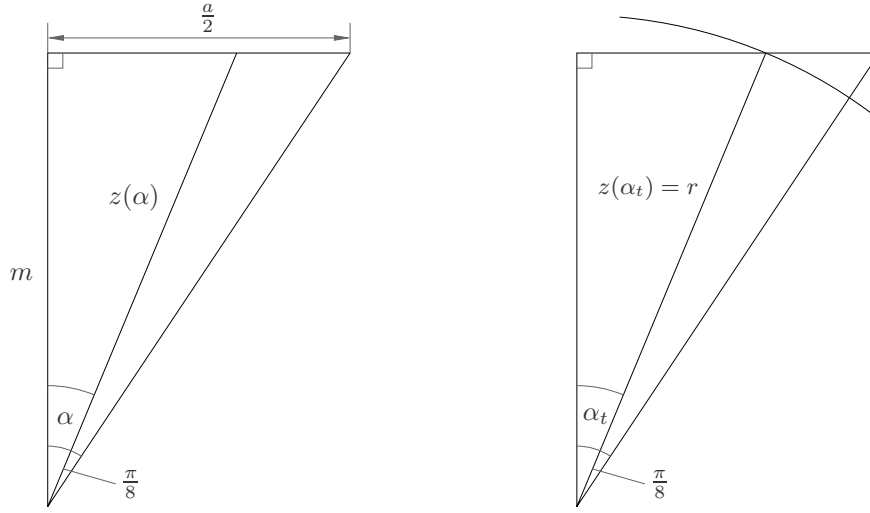


Figure 5: Curve based approximation

Least sum of distances

In this case, the elementary distance is not squared, thus we have to manually take into account, that between the threshold angle α_t and $\frac{\pi}{8}$ the distance is negative.

$$f(\alpha, r) := z_0(\alpha) - z(\alpha)$$

$$z_0(\alpha_t) = z(\alpha_t) \Rightarrow \alpha_t = \arccos\left(\frac{\sqrt{2} + 1}{2} \frac{a}{r}\right)$$

$$F(r) := \int_0^{\alpha_t} f(\alpha, r) d\alpha - \int_{\alpha_t}^{\frac{\pi}{8}} f(\alpha, r) d\alpha.$$

As a last step – just as before – we only need to solve the equation $F'(r) = 0$. We get

$$a = \frac{2}{\sqrt{2} + 1} \cos\left(\frac{\pi}{16}\right) r.$$

6 Construction

In this section we show how to construct an approximation. Let the radius r of the approximated circle be given. The approximation will be given by the number of used

1-steps and 2-steps. Depending on the approximation method one can compute the sidelength (a) of the octagon that is given in the previous section (and it also can be found in Table 4.) By Table 2 we can compute the number of 1's and 2's used in the neighbourhood sequence to obtain the desired octagon. The side length of the side b can be computed by Table 1.

On Fig. 6 and 7 the number of 1's and 2's of the best approximating neighbourhood sequences are given depending on the radius of the approximated circle.

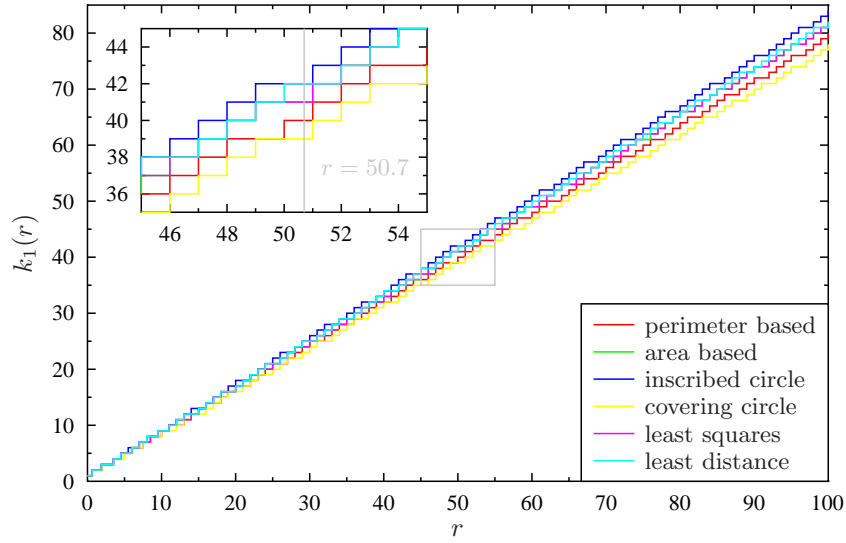


Figure 6: The number of 1's obtaining the best approximation

Now we show an example. Let $r = 50.7$. From the value of a the values of k_1 and k_2 are computed, then from k_1 the value of b can be calculated. Table 3 shows the best approximations using the inner octagon descriptor.

In Fig. 8 the best approximations obtained are shown.

7 Comparison

Here we summarize our main results in the form of a table. Table 4 shows the exact (continuous) sidelengths a calculated by using the methods described in the preceding section. r means the radius of the given circle. With the help of Table 2 we can get the number of values 1 and 2 in the neighbourhood sequence we look for, final formulae shown in Table 5. Then $k = k_1 + k_2$ gives the minimum length of the

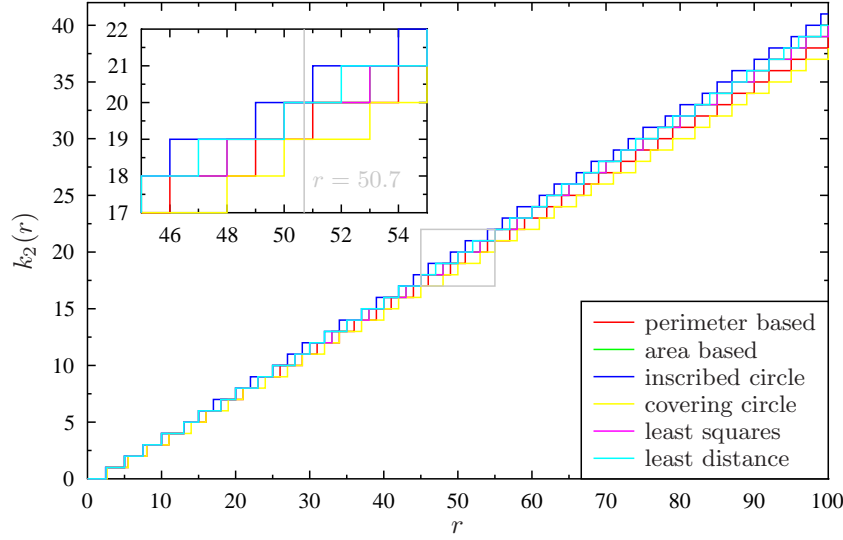


Figure 7: The number of 2's obtaining the best approximation

sequence. After the k th element the sequence may contain arbitrary values. The order of the first k elements is also arbitrary, since their permutation yields the same octagon. We also provide numerical results (rounded to 6 digits) to make the comparison of the coefficients easier. The developers only need to build Table 5 into their software supporting neighbourhood sequence based distance functions. In the table $[x]$ means the integer part of a real number x .

We can choose the used descriptor and approximation method depending on our aim. For example, if we would like to have an exact number of pixels from a given distance of a center point, we shall use pixel descriptors with perimeter based approximation. If we would like to clusterize a plane with neighbourhood sequence generated discs, and we need clusters with fixed area, we shall use a convex hull descriptor (it makes no real difference whether we use the inner or the outer octagon) with area based approximation. The other approximations can be useful in visual applications, and their results can be more easily forecast, since we often work with the radius and the curve of the shapes when we imagine geometric ideas.

	a	k_1	k_2	b
Perimeter based	40	28	20	39.5980
Area based	40	29	20	41.0122
Inscribed circle	42	30	21	42.4264
Covering circle	38	27	19	38.1838
Least squares	40	29	20	41.0122
Least distance	42	29	21	41.0122

Table 3. The sidelengths a and b and the values k_1, k_2 approximating circle with $r = 50.7$

	Exact sidelength (a)	Numerical sidelength
Perimeter based	$\frac{\pi}{4} r$	$0.785398 r$
Area based	$\sqrt{\frac{\pi}{2(1+\sqrt{2})}} r$	$0.806626 r$
Inscribed circle	$2(\sqrt{2} - 1) r$	$0.828427 r$
Covering circle	$\sqrt{2 - \sqrt{2}} r$	$0.765367 r$
Least squares	$\frac{\pi}{4(\sqrt{2}+1) \ln \tan(\frac{5\pi}{16})} r$	$0.806852 r$
Least distance	$\frac{2}{\sqrt{2}+1} \cos\left(\frac{\pi}{16}\right) r$	$0.812509 r$

Table 4. The sidelength a expressed by the radius r in various approximations

	Pixel based ($k_1 \geq 1$)	
	k_1	k_2
Perimeter based	$[0.785398r + 1]$	$[0.392699r - 0.5]$
Area based	$[0.806626r + 1]$	$[0.403313r - 0.5]$
Inscribed circle	$[0.828427r + 1]$	$[0.414214r - 0.5]$
Covering circle	$[0.765367r + 1]$	$[0.382684r - 0.5]$
Least squares	$[0.806852r + 1]$	$[0.403426r - 0.5]$
Least distance	$[0.812509r + 1]$	$[0.406255r - 0.5]$

	Inner octagon	
	k_1	k_2
Perimeter based	$[0.555360r]$	$[0.392699r]$
Area based	$[0.570371r]$	$[0.403313r]$
Inscribed circle	$[0.585786r]$	$[0.414214r]$
Covering circle	$[0.541196r]$	$[0.382684r]$
Least squares	$[0.570531r]$	$[0.403426r]$
Least distance	$[0.574531r]$	$[0.406255r]$

	Outer octagon	
	k_1	k_2
Perimeter based	$[0.555360r]$	$[0.392699r - 0.5]$
Area based	$[0.570371r]$	$[0.403313r - 0.5]$
Inscribed circle	$[0.585786r]$	$[0.414214r - 0.5]$
Covering circle	$[0.541196r]$	$[0.382684r - 0.5]$
Least squares	$[0.570531r]$	$[0.403426r - 0.5]$
Least distance	$[0.574531r]$	$[0.406255r - 0.5]$

Table 5. The numerical values of k_1, k_2 expressed by the radius r in various approximations

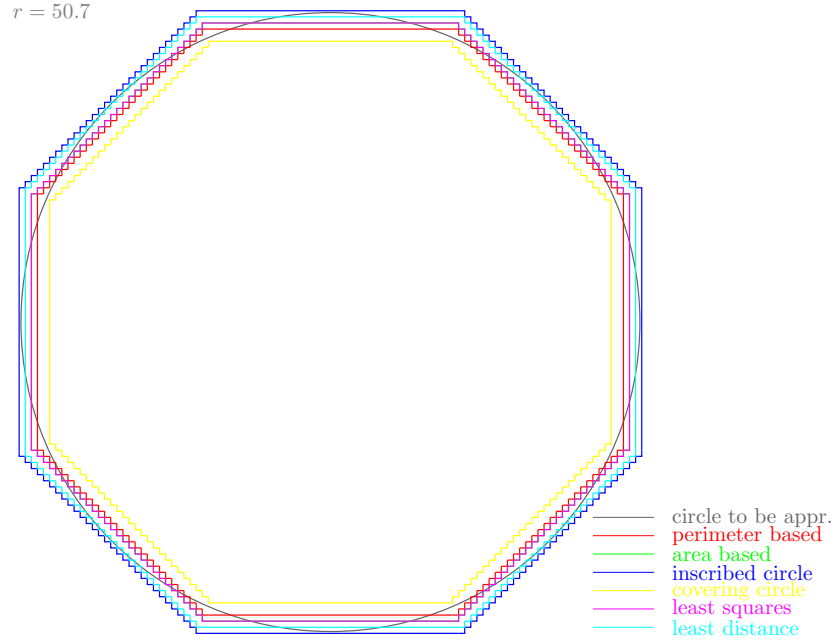


Figure 8: The best approximating digital circles of the Euclidean circle $r = 50.7$

8 Conclusions

The presented approximations work not only with integer radii (opposite to the previous papers, in which the best approximating neighbourhood sequence is computed to provide best approximation for the sequence of circles with integer radii). An example is shown for radius 50.7. Similarly, for any non-negative real radius one can compute the best approximation with our formulae. The digital discs and our formulae are useful for distance transforms, segmentation by colour clusterization and other algorithms in image processing. A discussion of a three dimensional approximation is provided in [5], but the extension of most approaches of the problem to three and more dimensions is a matter of future research. There are some related results on the triangular grid ([8, 9]) and in three dimensions on the face-centered and on the body-centered cubic grids ([10]).

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